Phase Space Reduction for Star-Products: An Explicit Construction for $\mathbb{C}P^n$

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Abstract

We derive a closed formula for a star-product on complex projective space and on the domain $SU(n+1)/S(U(1)\times U(n))$ using a completely elementary construction: Starting from the standard star-product of Wick type on $\mathbb{C}^{n+1}\setminus\{0\}$ and performing a quantum analogue of Marsden-Weinstein reduction, we can give an easy algebraic description of this star-product. Moreover, going over to a modified star-product on $\mathbb{C}^{n+1}\setminus\{0\}$, obtained by an equivalence transformation, this description can be even further simplified, allowing the explicit computation of a closed formula for the star-product on $\mathbb{C}P^n$ which can easily transferred to the domain $SU(n+1)/S(U(1)\times U(n))$.

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1 Introduction

The concept of deformation quantization has been defined by F. Bayen, M. Flato, C. Fronsdal, A. Lichnerowicz, D. Sternheimer in 1978 (cf. [2]) and in the more restricted context of Kähler manifolds by F. Berezin already in 1974 ([3]). The basic idea is to formally deform the pointwise commutative multiplication in the space of smooth complex-valued functions on a symplectic manifold M, $C^\infty(M)$, to a noncommutative associative multiplication, the star-product, whose first order commutator is proportional to the Poisson bracket. In addition, the definition of a star-product usually requires that the deformation series be local, i.e. the existence of an atlas in which each term in the series is locally a bidifferential operator, that the star-product with constant functions reduce to the pointwise multiplication, and that the complex conjugation be an antilinear involution in the deformed algebra, the most prominent cases being of Weyl-Moyal type where the formal parameter is considered to be purely imaginary and of Wick type where the formal parameter is taken to be real. The Wick-type case typically occurs on Kähler manifolds.

The nontrivial question of existence of star-products on every symplectic manifold has been positively answered by M. DeWilde and P.B.A. Lecomte in 1983 [9] using the computation of Hochschild and Chevalley-Eilenberg cohomology of $C^\infty(M)$ (e.g. [5]) and independently by B. Fedosov in 1985 ([10] and [11]) who did not use these cohomological computations.

However, closed formulas for the deformation series as in the case of the Weyl-Moyal product (resp. the Wick product) on a symplectic (resp. Kähler) manifold admitting a flat torsion-free symplectic (Kähler) connection (e.g [2]) are commonly regarded as practically nonexisting for all other manifolds with the following notable exceptions: the article of S. Gutt ([15]) for the cotangent bundle of a connected Lie group, the work of C. Moreno and P. Ortega-Navarro ([17], [18], [19]) who gave recursion formulas in the case of the 2-sphere, the Poincaré disk and their higher dimensional analogues, complex projective space and the symmetric hermitean domain $SU(1,n)/S(U(1)\times U(n))$ (e.g. [16], p.518 for details), and the work of M. Cahen, S. Gutt, J. Rawnsley ([8]) who gave an explicit description of the star-product of Berezin symbols in the case of the Poincaré disk.

Another problem which does not seem to have been much attacked in the framework of star products is the question whether and how the procedure of symplectic reduction—which has turned out to be a powerful tool in classical mechanics for both the construction of new symplectic manifolds and Liouville integrable systems—can be extended to a "reduction" of star products. For certain $U(1)$-actions this has been formulated in a rather general, but not too explicit fashion again by B. Fedosov ([12]). In the framework of geometric quantization of Kähler manifolds and coherent states (see the series [6], [7], [8] for details) the fact that the Kähler manifold is a $U(1)$ reduction of the total space (minus the zero-section) of its prequantum line bundle has been exploited

in [4] to reformulate the asymptotic limits already described in [7].

In this paper we give an explicit closed formula for a star-product on complex projective space and the domain $SU(1,n)/S(U(1)\times U(n))$ which in addition is shown to be the Marsden-Weinstein reduction of a star-product equivalent to the usual Wick product on the flat Kähler manifold $\mathbb{C}^{n+1}\setminus\{0\}$. The construction is completely elementary and does not use geometric quantization or deep cohomological results in complex differential geometry.

The paper is organized as follows: we show in section 2. that the usual Wick product of two functions on $\mathbb{C}^{n+1}\setminus\{0\}$ which are pulled back from complex projective space is a power series in pulled-back functions multiplied by "radial" functions (which only depend on the Euclidean distance from the origin). Moreover, the Wick product of two radial functions is radial but not simply pointwise multiplication. In section 3. we explicitly construct an equivalence transformation of the Wick product on $\mathbb{C}^{n+1}\setminus\{0\}$ thus obtaining a new star-product in which the radial functions behave like scalars in the subalgebra of $U(1)$-invariant functions. In section 4. we show that this new star-product on $\mathbb{C}^{n+1}\setminus\{0\}$ can simply be projected to complex projective space by essentially fixing the square of the Euclidean distance (which is—up to a constant factor—nothing but the $U(1)$ momentum map). The deformation series of this star-product on $\mathbb{C}\mathrm{P}^n$ can explicitly be written down (see Theorem 3.1). In Section 5. we show that all the results for $\mathbb{C}\mathrm{P}^n$ can easily be transferred to the noncompact domain $SU(1,n)/S(U(1)\times U(n))$. Finally, Section 6. contains a proof that our deformation series solves the recursion equations of the one implicitly given by Moreno in ([17]) and thus coincides with it.

2 Special properties of the Wick product

Let us first recall some well-known facts about the Wick-product on $\mathbb{C}^{n+1}\setminus\{0\}$: We denote by $z^0=q^0+\mathbf{i}p^0,\ldots,z^n=q^n+\mathbf{i}p^n$ the standard co-ordinates in $\mathbb{C}^{n+1}\setminus\{0\}$ and $\partial/\partial z^k=(1/2)(\partial/\partial q^k-\mathbf{i}\partial/\partial p^k)$ and $\partial/\partial\bar{z}^k=(1/2)(\partial/\partial q^k+\mathbf{i}\partial/\partial p^k)$ the corresponding complex differential operators. We shall use the summation convention, i.e the summation over repeated indices from zero to n is automatic. Furthermore, in this paper the symbol "z" will always designate a point in $\mathbb{C}^{n+1}\setminus\{0\}$ and for a smooth complex-valued function F on $\mathbb{C}^{n+1}\setminus\{0\}$ the notation $F(z)$ will *not* automatically imply that F is holomorphic unlike the traditional use in function theory. The standard symplectic form ω_0 on $\mathbb{C}^{n+1}\setminus\{0\}$ reads $\frac{\mathbf{i}}{2}dz^k\wedge d\bar{z}^k$. We shall denote by $C^\infty(\mathbb{C}^{n+1}\setminus\{0\})$ the space of smooth complex-valued functions on $\mathbb{C}^{n+1}\setminus\{0\}$. The Poisson bracket of two functions F and G in $C^\infty(\mathbb{C}^{n+1}\setminus\{0\})$ is given by

$$\{F,G\}=\frac{2}{\mathbf{i}}\Big(\frac{\partial F}{\partial z^k}\frac{\partial G}{\partial\bar{z}^k}-\frac{\partial F}{\partial\bar{z}^k}\frac{\partial G}{\partial z^k}\Big)$$

The *Wick product* is defined for two functions $F,G\in C^\infty(\mathbb{C}^{n+1}\setminus\{0\})$ by the

following formal series with formal parameter λ:

$$F * G := \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} \frac{\partial^r F}{\partial z^{i_1} \dots \partial z^{i_r}} \frac{\partial^r G}{\partial \bar{z}^{i_1} \dots \partial \bar{z}^{i_r}}. \tag{1}$$

It is well-known (cf. [3]) that this definition naturally extends to an *associative* product on the space of formal power series in λ having coefficients in $C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ which we shall call $\mathcal{A}$. This sum converges e.g. for complex-valued polynomials in which case we can identify λ with twice Planck's constant $\hbar$ yielding the first order commutator

$$F * G - G * F = \frac{\mathbf{i}\lambda}{2}\{F, G\} + \dots = \mathbf{i}\hbar\{F, G\} + \dots$$

In the following we consider functions $C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ of a particular form: Define $x : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{R}^+$ by

$$x(z) := \bar{z}^k z^k$$

From time to time we shall consider x as co-ordinate function along $\mathbb{R}^+$ on $\mathbb{C}^{n+1} \setminus \{0\} = \mathbb{R}^+ \times S^{2n+1}$. A function $R \in C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ is called *radial* iff there exists a C^∞-function $\varrho : \mathbb{R}^+ \to \mathbb{C}$ such that $R(z) = \varrho \circ x(z)$. The flow of the vector field $x\frac{\partial}{\partial x} = \frac{1}{2}(z^k \frac{\partial}{\partial z^k} + \bar{z}^k \frac{\partial}{\partial \bar{z}^k})$ consists of the scalar multiplication of a complex vector in $\mathbb{C}^{n+1} \setminus \{0\}$ by a positive real number. Furthermore, the flow of the vector field $Y(z) = \mathbf{i}(z^k \frac{\partial}{\partial z^k} - \bar{z}^k \frac{\partial}{\partial \bar{z}^k})$ generates scalar multiplication of z by an element of the unit circle $U(1)$. We call a function f in $C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ *homogeneous* iff it is invariant by the natural action of the group $\mathbb{C} \setminus \{0\}$ on $\mathbb{C}^{n+1} \setminus \{0\}$.

By direct calculation we find the following lemma:

Lemma: 2.1 *Let $R_1 = \varrho_2 \circ x$, $R_2 = \varrho_2 \circ x$, be radial functions, F be a $U(1)$-invariant function in $C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$, and f, g two homogeneous functions on $\mathbb{C}^{n+1} \setminus \{0\}$. Then the following relations for the Wick-product hold:*

i.)

$$R_1 * F = \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} x^r \frac{\partial^r \varrho_1}{\partial x^r}(x) \frac{\partial^r F}{\partial x^r} = F * R_1 \tag{2}$$

ii.) In particular we have

$$R_1 * R_2 = R_2 * R_1, \tag{3}$$

and $R_1 * R_2$ is again radial.

iii.) The Wick star-product of a radial with a homogeneous function reduces to pointwise multiplication:

$$R_1 * f = R_1 f = f * R_1. \tag{4}$$

iv.) For each positive integer $r \geq 0$ denote by M_r the following bidifferential operator defined on a pair of functions G and H in $C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$:

$$M_r(G, H) = x^r \frac{\partial^r G}{\partial z^{i_1} \cdots \partial z^{i_r}} \frac{\partial^r H}{\partial \bar{z}^{i_1} \cdots \partial \bar{z}^{i_r}}. \tag{5}$$

Then each function $M_r(f, g)$ is homogeneous and we have the obvious formula

$$f * g = \sum_{r=0}^{\infty} \frac{1}{r!} \frac{\lambda^r}{x^r} M_r(f, g). \tag{6}$$

Proof: Since radial and homogeneous functions are $U(1)$-invariant, and $\partial/\partial x$ vanishes on homogeneous functions only the first and the last part of this lemma need to be proved: denoting by E and $\bar{E}$ the Euler operators $z^i \frac{\partial}{\partial z^i}$ and $\bar{z}^i \frac{\partial}{\partial \bar{z}^i}$ we get by induction

$$z^{i_1} \ldots z^{i_r} \frac{\partial^r F}{\partial z^{i_1} \ldots \partial z^{i_r}}(z) = (\prod_{k=0}^{r-1}(E - k)F)(z) = (\prod_{k=0}^{r-1}(x\frac{\partial}{\partial x} - k)F)(z) = (x^r \frac{\partial^r F}{\partial x^r})(z)$$

where we have used that $x\frac{\partial}{\partial x} = (1/2)(E + \bar{E})$ and that $Y = \mathbf{i}(E - \bar{E})$ vanishes on $U(1)$-invariant smooth functions. In an analogous way the corresponding statement involving the $\bar{z}^i$ and $\partial/\partial\bar{z}^i$ is shown. This proves part i). For the part iv), note that a smooth complex-valued function is homogeneous iff it is annihilated by the two Euler operators E and $\bar{E}$. Clearly, $[E, \partial/\partial z^k] = -\partial/\partial z^k$ and $[\bar{E}, \partial/\partial \bar{z}^k] = -\partial/\partial \bar{z}^k$ whereas $[E, z^k] = z^k$ and $[\bar{E}, \bar{z}^k] = \bar{z}^k$ and the homogeneity of $M_r(f, g)$ follows by induction. ■

Consider now the complex projective space $\mathbb{C}\mathrm{P}^n$ with its natural projection $\pi : \mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{C}\mathrm{P}^n : z \mapsto [z]$ where $[z]$ denotes the complex line defined by z. $\mathbb{C}\mathrm{P}^n$ is a Kähler manifold with a complex structure inherited by $\mathbb{C}^{n+1} \setminus \{0\}$ and a symplectic form ω which can be defined by its pull-back $\pi^*\omega = \frac{\mathbf{i}}{2}\partial\bar{\partial}\log x$ (see e.g. [14] for details) or by phase space reduction, see section 4. The pull-back of a smooth complex-valued function ϕ to $\mathbb{C}^{n+1} \setminus \{0\}$, $\pi^*\phi$ is obviously homogeneous, and vice versa.

Now looking at the third and forth statement of the above lemma one might be tempted to directly project the above Wick star-product of two homogeneous functions to a star-product on functions on complex projective space by using λ/x in some sense as a new formal parameter. However, by the first statement of the above lemma it quickly becomes clear that things are not that simple: the Wick product of two radial functions is *not* pointwise multiplication, although the Wick product of a radial and a homogeneous function is pointwise. This fact destroys the associativity of such a naively defined "star-product". But we shall see in the next section that an equivalence transformation modifying the Wick star-product a little will lead out of this difficulty.

3 Equivalence transformation of the Wick-product

The aim of this section is to find an equivalence transformation S (see e.g. [2], part I, p.85 for definitions) of the Wick-product $*$ into a new star-product $\tilde{*}$ with the property that the $\tilde{*}$-product of two radial functions is the ordinary pointwise product. By eqn. (2) und eqn. (3) we only have to consider the associative local deformation of the pointwise product of two smooth complex-valued functions ϱ_1, ϱ_2 on the positive real line defined by

$$\varrho_1 \star \varrho_2(x) := \sum_{r=0}^{\infty} \lambda^r \frac{x^r}{r!} \frac{\partial^r \varrho_1}{\partial x^r}(x) \frac{\partial^r \varrho_2}{\partial x^r}(x). \tag{7}$$

For general reasons, (cf. e.g. [13, p. 46]) this deformation must be equivalent to the pointwise multiplication because the second local Hochschild cohomology group of the associative algebra of smooth complex-valued functions on $\mathbb{R}^+$ vanishes being isomorphic to $\Gamma(\bigwedge^2 T\mathbb{R}^+) = \{0\}$ (cf. e.g. [5]).

Such an equivalence transformation S is a formal power series $S(x, \partial_x) = \sum_{r=0}^{\infty} \lambda^r S_r(x, \partial_x)$ (where the S_r are differential operators acting on the space of all smooth complex-valued functions on the positive real line and we have used the notation $\partial_x := \partial/\partial x$) with the property

$$S(\varrho_1 \star \varrho_2) = (S\varrho_1)(S\varrho_2). \tag{8}$$

for arbitrary smooth functions $\varrho_1, \varrho_2 : \mathbb{R}^+ \to \mathbb{C}$ and S_0 is equal to the identity. In order to perform concrete calculations it will turn out to be convenient to work with the following *symbol* $\hat{S}$ of S which will be a power series in λ whose coefficients are contained in the space of smooth complex-valued functions on $\mathbb{R}^+ \times \mathbb{R}$ that are polynomial in the second variable and which is defined by

$$\hat{S}(x, \alpha) e_\alpha(x) := (S e_\alpha)(x)$$

where e_α denotes the exponential function $x \mapsto e^{\alpha x}$ on the real line for $\alpha \in \mathbb{R}$. This makes sense since $\partial_x^r e_\alpha = \alpha^r e_\alpha \quad \forall r \in \mathbb{N}$. Moreover, since S is uniquely determined by its action on the monomials x^r it can obviously be regained from its symbol $\hat{S}$ by putting the monomials in the second variable α occurring in $\hat{S}(x, \alpha)$ on the right and then substituting α by ∂_x (the standard ordering prescription).

We get the following theorem:

Theorem: 3.1 *The deformation $\star$ is equivalent to the pointwise product and every equivalence transformation S satisfying eqn. (8) has a symbol of the form*

$$\hat{S}(x, \alpha) = \exp\left(\frac{x}{\lambda}\left(D(x, \lambda)\log(1 + \lambda\alpha) - \lambda\alpha\right)\right) \tag{9}$$

where $D(x, \lambda) := exp(\lambda C(x, \lambda))$ is an arbitrary formal power series in λ with smooth coefficient functions starting with 1. *Moreover $S1 = 1$ for the constant function* 1.

Proof: First note that in terms of exponential functions e_α, e_β with $\alpha, \beta \in \mathbb{R}$ the above star-product eqn. (7) on the positive real line has the following particularly simple form:

$$e_\alpha \star e_\beta = e_{\alpha+\beta+\lambda\alpha\beta}$$

where the r.h.s. is well-defined as a formal series in λ. Using this equation the condition eqn. (8) can be reformulated into the following functional equation for the symbol of S:

$$\hat{S}(x, \lambda\alpha\beta + \alpha + \beta)e^{\lambda\alpha\beta x} = \hat{S}(x, \alpha)\hat{S}(x, \beta) \tag{10}$$

Since $\hat{S}_0 = 1$ we may take the formal logarithm $\lambda T = \log \hat{S}$ on both sides thus getting the following equation for T, which again is a formal power series starting with a constant:

$$\lambda\alpha\beta x + \lambda T(x, \lambda\alpha\beta + \alpha + \beta) = \lambda T(x, \alpha) + \lambda T(x, \beta)$$

By differentiating this equation with respect to α and β we see that every solution of this functional equation satisfies the following differential equation, where $'$ denotes the differentiation with respect to the second argument.

$$x + T''(x, \lambda\alpha\beta + \alpha + \beta)(1 + \lambda(\lambda\alpha\beta + \alpha + \beta)) + \lambda T'(x, \lambda\alpha\beta + \alpha + \beta) = 0$$

With $\gamma := \lambda\alpha\beta + \alpha + \beta$ this differential equation has the general solution

$$T(x, \gamma) = \frac{x}{\lambda^2} e^{\lambda C(x,\lambda)} \log(1 + \lambda\gamma) - \frac{x\gamma}{\lambda} + \frac{xB(x, \lambda)}{\lambda}$$

depending on two arbitrary integration constants $B(x, \lambda)$ and $C(x, \lambda)$. Now we can easily check that this general solution of the differential equation leads to a solution $\hat{S}$ of the functional equation (10) iff B vanishes identically. In this case we have $\hat{S}_0 = 1$. The formal power series $C(x, \lambda) = \sum_{r=0}^\infty \lambda^r C_r(x)$ is completely arbitrary. Since $(S1)(x)$ is obviously equal to $\hat{S}(x, 0)$ which in turn is equal to 1 the theorem is proved. ■

In the next step we regard the differential operator ∂_x as a differential operator on $C^\infty(\mathbb{C}^{n+1} \setminus \{0\}) \cong C^\infty(\mathbb{R}^+ \times S^{2n+1})$ by observing that $\partial_x = \frac{1}{2x}(z^k \frac{\partial}{\partial z^k} + \bar{z}^k \frac{\partial}{\partial \bar{z}^k})$. Then S extends to a formal power series in λ with differential operators acting in $C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ which we shall also denote by S. We can now use S to transform the Wick-product $*$ into a new star-product $\tilde{*}$ in $\mathcal{A}$ defined by

$$F \,\tilde{*}\, G := S\left(\left(S^{-1}F\right) * \left(S^{-1}G\right)\right) \tag{11}$$

for two functions $F, G \in \mathcal{A}$. If $f = \phi \circ \pi$ is a homogeneous function we have $\partial_x f = 0$. This leads to

$$Sf = S^{-1}f = f$$

Combining this result with the properties of the Wick-product in lemma (2), (4) and (6) we get four relations for the new star-product analogous to those mentioned in the lemma in the previous section:

Theorem: 3.2 *Let $R_1 = \varrho_1 \circ x$, $R_2 = \varrho_2 \circ x$ be two radial functions, $f = \phi \circ \pi$, $g = \psi \circ \pi$ two homogeneous functions and F a $U(1)$-invariant function in $C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$. Then*

$$\begin{aligned} R_1 \tilde{*} R_2 &= R_1 R_2 = R_2 \tilde{*} R_1 \\ R_1 \tilde{*} F &= R_1 F = F \tilde{*} R_1 \\ R_1 \tilde{*} f &= R_1 f = f \tilde{*} R_1 \\ f \tilde{*} g = S(f * g) &= \sum_{r=0}^{\infty} \frac{1}{r!} \left(S(x, \partial_x) \left(\frac{\lambda}{x} \right)^r \right) M_r(f, g). \end{aligned} \tag{12}$$

Because $M_r(f, g)$ is again homogeneous, the differential operators in S act on the radial functions $(\frac{\lambda}{x})^r$ only. We can give an explicit form of these terms. First we use the equation (9) for S to find the following proposition.

Proposition: 3.3 *For any formal power series $D(x, \lambda)$ starting with 1 the following equations hold for the differential operator $S(x, \partial_x)$ defined by its symbol as in eqn. (9):*

$$S(x, \partial_x) x = D(x, \lambda) x \tag{13}$$

For $r \in \mathbb{N}$ we have

$$S(x, \partial_x) x^r = (D(x, \lambda) x)^r \prod_{k=0}^{r-1} \left(1 - k \frac{\lambda}{D(x, \lambda) x} \right)$$

$$S(x,\partial_x) \frac{1}{x^r} = \left(\frac{1}{D(x, \lambda) x} \right)^r \prod_{k=0}^{r} \left(1 + k \frac{\lambda}{D(x, \lambda) x} \right)^{-1} = \left(\frac{1}{D(x, \lambda) x} \right)^r \sum_{s=0}^{\infty} \left(\frac{\lambda}{D(x, \lambda) x} \right)^s A_s^{(r)} \tag{14}$$

where the $A_s^{(r)}$ are rational numbers defined by $A_0^{(0)} = 1$, $A_s^{(0)} = 0 \;\; \forall s \geq 1$, and for $s \geq 0$, $r \geq 1$

$$A_s^{(r)} := \frac{1}{(r-1)!} \sum_{k=1}^{r} \binom{r-1}{k-1} k^{s+r-1} (-1)^{r+s-k}. \tag{15}$$

Proof: The first equation is proved by observing that $S(x, \partial_x) x = (\partial/\partial\alpha)(S e_\alpha)(x)|_{\alpha=0}$ and using the symbol $\hat{S}$. For the second and the third statement we use the fact that the Wick-product of x and x^r is $x^{r+1} + \lambda r x^r$. We use eqn. (11) to obtain a recursion formula for $S x^r$ and $S x^{-r}$. These recursion formulas can be solved using eqn. (13). The second equation in eqn. (14) is proved by performing a partial fraction decomposition of the product and induction. ■

We want to choose the formal series D in such a way that in the new star-product of two homogeneous functions of λ only the combination $(\frac{\lambda}{x})^r$ times

some homogeneous term depending on f, g occurs in each order. This is the case iff the series D is of the form

$$D(x,\lambda) = \sum_{r=0}^{\infty} \left(\frac{\lambda}{x}\right)^r d_r \qquad \text{where } d_0 = 1 \text{ and } d_r \in \mathbb{C}. \tag{16}$$

We shall henceforth assume that D is of this special form with arbitrary complex numbers d_r. Then we get the following formula for the new star-product of two homogeneous functions f and g:

$$f \tilde{*} g = \sum_{r=0}^{\infty} \frac{1}{r!} \left(\frac{\lambda}{D(x,\lambda)x}\right)^r \prod_{k=1}^{r} \left(1 + k\frac{\lambda}{D(x,\lambda)x}\right)^{-1} M_r(f,g) \tag{17}$$

where the empty product in the case $r = 0$ is defined to be 1. This can be rewritten as

$$f \tilde{*} g = \sum_{r=0}^{\infty} \left(\frac{\lambda}{x}\right)^r K_r(f,g) \tag{18}$$

where the $K_r(f,g)$ are some linear combination — depending on the choice of the series D — of the $M_s(f,g)$ with $s \le r$. In particular $K_r(f,g)$ is again homogeneous.

An interesting property of the original Wick product is the fact that it can be expressed as a power series in a differential operator on smooth complex-valued functions F on $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{n+1} \setminus \{0\}$ namely the operator $\mathcal{P}$ defined by

$$\mathcal{P}F := \frac{\partial^2 F}{\partial z^i \partial \bar{w}^i}.$$

Upon writing m for the evaluation of a smooth complex-valued function F on $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{n+1} \setminus \{0\}$ on the diagonal (i.e $m(F)(z) := F(z,w)|_{z=w}$) we get the following formula for the old Wick product of G and H in $\mathcal{A}$:

$$G * H = m e^{\lambda \mathcal{P}} G \otimes H.$$

A similar thing can be done for the new star-product: We define for a smooth complex-valued function F on $\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{n+1} \setminus \{0\}$ the following differential operators:

$$N(F)(z,w) := z^i \bar{w}^i \frac{\partial^2 F}{\partial z^j \partial \bar{w}^j}(z,w) \tag{19}$$

and

$$\mathcal{M}_r(F)(z,w) := z^{i_1} \ldots z^{i_r} \bar{w}^{i_1} \ldots \bar{w}^{i_r} \frac{\partial^{2r} F}{\partial z^{j_1} \ldots \partial z^{j_r} \partial \bar{w}^{j_1} \ldots \bar{w}^{j_r}}(z,w). \tag{20}$$

We clearly get $M_1(F,G) = m \circ \mathcal{M}_1(F \otimes G) = m \circ N(F \otimes G)$ and $M_r(F,G) = m \circ \mathcal{M}_r(F \otimes G)$. By induction we find the following recursion formula

$$\mathcal{M}_{r+1} = (N - r(n-r) - rH)\mathcal{M}_r$$

where $H := z^i \frac{\partial}{\partial z^i} + \bar{z}^i \frac{\partial}{\partial \bar{z}^i} + w^i \frac{\partial}{\partial w^i} + \bar{w}^i \frac{\partial}{\partial \bar{w}^i}$. Note that this operator vanishes on smooth complex-valued functions F which are doubly homogeneous, i.e. $F(\alpha z, \beta w) = F(z, w)$, and commutes with $\mathcal{M}_r$ and N. This recursion can be solved and leads to the equation

$$\mathcal{M}_r = \prod_{s=0}^{r-1} (N - s(n-s) - sH) \tag{21}$$

Therefore all the operators $\mathcal{M}_r$ are polynomials in the operator N and H.

4 The star-product on $\mathbb{C}\mathrm{P}^n$ and phase space reduction

Before we construct the star-product on $\mathbb{C}\mathrm{P}^n$ let us first recall some well-known facts about the phase space reduction of $\mathbb{C}^{n+1} \setminus \{0\}$ to $\mathbb{C}\mathrm{P}^n$ with respect to the $U(1)$-action $z \mapsto e^{\mathbf{i}\phi} z$, $\phi \in \mathbb{R}$. We basically use the notation of [1]. An ad^*-equivariant momentum mapping is given by

$$J(z) := -\frac{1}{2} x$$

and every $\mu \in \mathbb{R}^-$ is a regular value of J. Therefore $J^{-1}(\mu)$ is a submanifold of $\mathbb{C}^{n+1} \setminus \{0\}$, the $(2n+1)$-sphere of radius $\sqrt{-2\mu}$. The reduced phase space is then given by $J^{-1}(\mu)/U(1) \cong \mathbb{C}\mathrm{P}^n$. A symplectic form on $\mathbb{C}\mathrm{P}^n$ is uniquely defined by

$$i_\mu^* \omega_0 = \pi_\mu^* \omega_\mu$$

where $i_\mu : J^{-1}(\mu) \to \mathbb{C}^{n+1} \setminus \{0\}$ is the inclusion map, and $\pi_\mu : J^{-1}(\mu) \to J^{-1}(\mu)/U(1)$ the projection on the equivalence classes. For a $U(1)$-invariant function $F \in C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ we can define the reduced function $F_\mu : \mathbb{C}\mathrm{P}^n \to \mathbb{C}$ by

$$F_\mu([z]) := F \circ i_\mu(z) \qquad \text{for } z \in J^{-1}(\mu)$$

Note that F is $U(1)$-invariant iff $\{F, J\} = 0$. For two $U(1)$-invariant functions F, G we have

$$(\{F, G\})_\mu = \{F_\mu, G_\mu\}_\mu$$

where $\{,\}_\mu$ denotes the Poisson bracket on $\mathbb{C}\mathrm{P}^n$ which is defined by ω_μ.

In the next Proposition we show how phase space reduction can be extended to the star-products on $\mathbb{C}^{n+1} \setminus \{0\}$:

Proposition: 4.1 *Denote by $\mathcal{A}^0$ the subspace of $\mathcal{A}$ of those power series whose coefficients are $U(1)$-invariant. Suppose in addition that the power series D eqn. (16) is equal to* 1. *Then*

i.) *A function F in $C^\infty(\mathbb{C}^{n+1}\setminus\{0\})$ is $U(1)$-invariant iff $F*J-J*F=0$ whence $\mathcal{A}^0$ is both a $*$- and a $\tilde{*}$-subalgebra of $\mathcal{A}$.*

ii.) *Fix a negative real number μ. Denote by $\mathcal{J}_\mu$ and $\tilde{\mathcal{J}}_\mu$ the $*$-ideal and the $\tilde{*}$-ideal of $\mathcal{A}^0$ generated by $J-\mu$, respectively. Then $\tilde{\mathcal{J}}_\mu$ consists of those power series in $\mathcal{A}^0$ whose coefficients vanish on the $2n+1$-dimensional sphere $J^{-1}(\mu)$ and $\mathcal{J}_\mu$ is equal to $S^{-1}\tilde{\mathcal{J}}_\mu$.*

iii.) *Denote by $\mathcal{B}$ (and by $\pi^*\mathcal{B}$) the space of formal power series with coefficients in $C^\infty(\mathbb{C}P^n)$ $(\pi^*(C^\infty(\mathbb{C}P^n)))$. Then we have the following direct sums:*

$$\mathcal{A}^0=\pi^*\mathcal{B}\oplus\mathcal{J}_\mu=\pi^*\mathcal{B}\oplus\tilde{\mathcal{J}}_\mu \tag{22}$$

Proof: 1. Using the definition of the Wick product eqn. (1) we see that $F*J-J*F=\frac{\mathrm{i}\lambda}{2}\{F,J\}$, and the latter vanishes iff F is $U(1)$-invariant. In any associative algebra the subspace of all elements commuting with a fixed one is a subalgebra. Since S leaves $\mathcal{A}^0$ invariant (because S commutes with the $U(1)$-action) it follows that $\mathcal{A}^0$ is also a $\tilde{*}$-subalgebra of $\mathcal{A}$.

2. Let g be a $U(1)$-invariant function vanishing on the $2n+1$-dimensional sphere $J^{-1}(\mu)$. Using the diffeomorphism $\mathbb{C}^{n+1}\setminus\{0\}\cong\mathbb{R}\times J^{-1}(\mu)$ and the $U(1)$-invariance of g this function can be written as $g(z)=h(x,[z])$ where h is a smooth complex-valued function on $\mathbb{R}\times\mathbb{C}\mathrm{P}^n$. Then by Hadamard's trick

$$\begin{aligned} h(x,[z]) &= h(x,[z])-h(-2\mu,[z])=\int_0^1 dt\frac{d}{dt}h(tx+(1-t)(-2\mu),[z]) \\ &= -2(\int_0^1 dt(\partial_1 h)(tx+(1-t)(-2\mu),[z]))(J(x)-\mu) \end{aligned}$$

which—by the obvious $U(1)$-invariance of the integral term—shows that g is a $U(1)$-invariant multiple of $J-\mu$. But since $(J-\mu)g=(J-\mu)\,\tilde{*}\,g=g\,\tilde{*}\,(J-\mu)$ by eqn. (12) we obtain the asserted characterization of the $\tilde{*}$-ideal $\tilde{\mathcal{J}}_\mu$. Moreover, because of the choice $D(x,\lambda)=1$ and eqn. (13) we get

$$\mathcal{J}_\mu=(J-\mu)*\mathcal{A}^0=S^{-1}(S(J-\mu)\,\tilde{*}\,S\mathcal{A}^0)=S^{-1}((J-\mu)\mathcal{A}^0)=S^{-1}\tilde{\mathcal{J}}_\mu.$$

3. Let g be a smooth complex-valued $U(1)$-invariant function on $\mathbb{C}^{n+1}\setminus\{0\}$. Obviously $g=\pi^*g_\mu+(g-\pi^*g_\mu)$ where $g-\pi^*g_\mu$ clearly vanishes on $J^{-1}(\mu)$. Furthermore π^*g vanishes on $J^{-1}(\mu)$ iff g vanishes on $J^{-1}(\mu)$. This shows the first direct sum decomposition in eqn. (22). Applying the linear bijection S^{-1} to this first direct sum, observing that $Sf=f=S^{-1}f$ for all $f\in\pi^*\mathcal{B}$ and using part 2 of this proposition we have shown the second direct sum in eqn. (22). ■

Mapping homogeneous functions to homogeneous functions, the bidifferential operators M_r (eqn. (5)) induce corresponding bidifferential operators on $C^\infty(\mathbb{C}\mathrm{P}^n)\times C^\infty(\mathbb{C}\mathrm{P}^n)$ by (recall that $x:=\bar{z}^i z^i$):

$$\tilde{M}_r(\phi,\psi)([z]):=M_r(\phi\circ\pi,\psi\circ\pi)(z)=x^r\frac{\partial^r(\phi\circ\pi)}{\partial z^{i_1}\cdots\partial z^{i_r}}(z)\frac{\partial^r(\psi\circ\pi)}{\partial\bar{z}^{i_1}\cdots\partial\bar{z}^{i_r}}(z). \tag{23}$$

where ϕ, ψ are two smooth complex-valued functions on CP^n. In a completely analogous way, we may construct bidifferential operators $\tilde{K}_r$ on $C^\infty(\mathbb{C}\mathrm{P}^n) \times C^\infty(\mathbb{C}\mathrm{P}^n)$ and differential operators $\tilde{N}$ and $\tilde{\mathcal{M}}_r$ on $C^\infty(\mathbb{C}\mathrm{P}^n \times \mathbb{C}\mathrm{P}^n)$ from the bidifferential operators K_r (eqn. (18)) on $C^\infty(\mathbb{C}^{n+1} \setminus \{0\}) \times C^\infty(\mathbb{C}^{n+1} \setminus \{0\})$ and the differential operators $N, \mathcal{M}_r$ (eqn. (19), eqn. (20)) on $C^\infty(\mathbb{C}^{n+1} \setminus \{0\} \times \mathbb{C}^{n+1} \setminus \{0\})$, respectively. The polynomial relation eqn. (21) also holds for $\tilde{N}$ and $\tilde{\mathcal{M}}_r$ after replacing H by zero.

We can now state the main theorem:

Theorem: 4.2 *Let $\phi, \psi : \mathbb{C}P^n \to \mathbb{C}$ be two C^∞-functions, let the bidifferential operators $\tilde{M}_r$ defined as is eqn. (23), let F and G two smooth complex-valued $U(1)$-invariant functions on $\mathbb{C}^{n+1} \setminus \{0\}$, and μ a negative real number. Suppose that the series $D(x, \lambda)$ is equal to 1. Then*

i.)

$$\begin{aligned} \phi *^\mu \psi &:= \phi\psi + \sum_{r=1}^{\infty} \left(\frac{\lambda}{-2\mu}\right)^r \sum_{s=1}^{r} \sum_{k=1}^{s} \frac{1}{s!(s-k)!(k-1)!} k^{r-1} (-1)^{r-k} \, \tilde{M}_s(\phi, \psi) \\ &=: \sum_{r=0}^{\infty} \lambda^r \frac{1}{(-2\mu)^r} \tilde{K}_r(\phi, \psi). \end{aligned} \tag{24}$$

defines a star-product with first order commutator $\frac{\mathrm{i}}{2}\{\phi, \psi\}_\mu$.

ii.) Let F, G be two $U(1)$-invariant functions then we have

$$\left(F \,\tilde{*}\, G\right)_\mu = F_\mu *^\mu G_\mu.$$

iii.) There are the following isomorphisms of associative algebras:

$$(\mathcal{B}, *^\mu) \cong (\mathcal{A}^0, \tilde{*})/\tilde{\mathcal{J}}_\mu \cong (\mathcal{A}^0, *)/\mathcal{J}_\mu$$

Proof: 1. This can either be computed directly by using the fact that the $\tilde{*}$ product of a radial function with a $U(1)$-invariant one is simply pointwise multiplication. A more direct way of seeing this comes from the above proposition: Since $\mathcal{B} \cong \mathcal{A}^0/\tilde{\mathcal{J}}_\mu$ as vector spaces and $\tilde{\mathcal{J}}_\mu$ is a $\tilde{*}$-ideal of $\mathcal{A}^0$ the space $\mathcal{B}$ is equipped with an associative multiplication induced by $\tilde{*}$ where "mod $\tilde{\mathcal{J}}_\mu$" is translated into "set x equal to -2μ". Moreover, since $*$ and $\tilde{*}$ are equivalent and the pointwise multiplication is commutative it is easy to see that $*$ and $\tilde{*}$ produce the same first order commutator.

2. and 3. This is immediate from the above proposition and the first part of this theorem since the linear map $F \mapsto \pi^* F_\mu$ is nothing but the projection onto $\pi^*\mathcal{B}$ along the $\tilde{*}$-ideal $\tilde{\mathcal{J}}_\mu$. ■

Remarks:

i.) In principle we could have obtained the star-product on $\mathbb{C}\mathrm{P}^n$ directly from the Wick product by using the above second direct sum decomposition of $\mathcal{A}^0$ into $\pi^*\mathcal{B}$ and the $*$-ideal $\mathcal{J}_\mu$: for two smooth complex-valued functions ϕ and ψ on $\mathbb{C}\mathrm{P}^n$ we can form the Wick product $\pi^*\phi * \pi^*\psi$ and recursively seperate off order by order $*$-factors of $J-\mu$.

ii.) For $D \not\equiv 1$ we still can construct a star-product on $\mathbb{C}\mathrm{P}^n$ from the star-product $\tilde{*}$ on $\mathbb{C}^{n+1}\setminus\{0\}$. Equation (17) implies that the star-product constructed in this way is obtained from the star-product for $D \equiv 1$ by a "reparametrisation" of the formal parameter: $\lambda \mapsto \lambda/(-2\mu D(-2\mu,\lambda))$. Such a substitution is algebraically well defined because μ is just a real number, and the formal power series $D(-2\mu,\lambda)$ starts with 1, and obviously yields a star-product again.

iii.) An immediate consequence of this theorem is the fact that for $D \equiv 1$ the $\tilde{*}$-commutator with J coincides with the classical Poisson-bracket and hence generates the same $U(1)$-action. Thus, we may interpret J for this choice of D as a "quantum momentum mapping". In the general case, we can proceed as follows: Using the facts that the Wick-commutator of an arbitrary function F and J is just the Poisson bracket times $\frac{\mathbf{i}\lambda}{2}$ and has no higher orders in λ, and that the action of $\{\cdot, J\}$ on F is a derivation which commutes with ∂_x and therefore with S and S^{-1}, we conclude:

$$F \,\tilde{*}\, SJ - SJ \,\tilde{*}\, F = \frac{\mathbf{i}\lambda}{2}\{F, J\}$$

This means that $SJ = DJ$ is the quantum momentum mapping. Hence, the case $D \equiv 1$ is distinguished by the fact that the classical and quantum momentum maps coincide.

5 The domain $SU(1,n)/S(U(1)\times U(n))$

In this section we shall sketch a modification of the above results to the non-compact dual (in the sense of Riemannian symmetric spaces [16]) of complex projective space: consider again $\mathbb{C}^{n+1}\setminus\{0\}$ as a complex manifold. Let g^{ij} be a diagonal matrix with $(g^{ij}) = diag(-1, 1, \ldots, 1)$. Consider the modified symplectic form $\omega_1 := \frac{\mathbf{i}}{2} g^{ij} dz^i \wedge d\bar{z}^j$ and its corresponding Poisson bracket. The modified Wick product $*_1$ is then given by

$$F *_1 G := \sum_{r=0}^{\infty} \frac{\lambda^r}{r!} g^{i_1 j_1} \cdots g^{i_r j_r} \frac{\partial^r F}{\partial z^{i_1} \ldots \partial z^{i_r}} \frac{\partial^r G}{\partial \bar{z}^{j_1} \ldots \partial \bar{z}^{j_r}}.$$

This is easily seen to be an associative star-product corresponding to the modified symplectic form ω_1. Let $y(z) := g^{ij} z^i \bar{z}^j$. Consider the open subset

$C_+^{n+1} \subset \mathbb{C}^{n+1} \setminus \{0\}$ defined by

$$C_+^{n+1} := \{z \in \mathbb{C}^{n+1} \setminus \{0\} | y(z) > 0\}.$$

It is clear that for each $z \in C_+^{n+1}$ the complex ray through z is also contained in C_+^{n+1}. Define

$$D^n := \pi(C_+^{n+1}) \ \subset \ \mathbb{C}\mathrm{P}^n.$$

D^n becomes a Kähler manifold with complex structure induced by that of $\mathbb{C}\mathrm{P}^n$ and symplectic form derived from the Kähler potential $\log y$. The noncompact semisimple Lie group $SU(1,n)$ acting canonically on $\mathbb{C}^{n+1} \setminus \{0\}$ obviously preserves the function y and hence the open subset C_+^{n+1}. It is easy to check that $SU(1,n)$ acts transitively on D^n via its action on the complex rays in C_+^{n+1} thereby leaving the Kähler structure of D^n invariant. The isotropy subgroup of the standard ray through $(1,0,\ldots,0)^T$ is clearly given by $S(U(1) \times U(n))$ hence D^n is isomorphic to the irreducible hermitean symmetric space $SU(1,n)/S(U(1) \times U(n))$ (see [16], p. 518).

It is not difficult to see that the whole construction mentioned in the previous section can be transferred to D^n by substituting $\mathbb{C}^{n+1} \setminus \{0\}$ by C_+^{n+1}, homogeneous functions on $\mathbb{C}^{n+1} \setminus \{0\}$ by homogeneous functions merely defined on C_+^{n+1}, the function x by y, radial functions by functions of y and the bidifferential operators M_r by

$$\check{M}_r(G,H) := y^r g^{i_1 j_1} \cdots g^{i_r j_r} \frac{\partial^r G}{\partial z^{i_1} \cdots \partial z^{i_r}} \frac{\partial^r H}{\partial \bar{z}^{j_1} \cdots \partial \bar{z}^{j_r}}.$$

Observing that $J_1(z) := -\frac{1}{2} y(z)$ is a momentum map for the canonical $U(1)$-action on C_+^{n+1} corresponding to the symplectic form ω_1 we see that D^n is a reduced phase space of C_+^{n+1}. Consequently, both proposition (4.1) and theorem (4.2) (with its explicit formula eqn. (24)) are also valid for the domain D^n when modified along the lines given above.

6 Comparison with existing results

In this section we want to prove that in the case of $\mathbb{C}\mathrm{P}^n \cong S^2$ our star-product $*^\mu$ for $\mu = \frac{1}{2}$ is the same as the star product constructed by Moreno and Ortega-Navarro in [17] iff $D \equiv 1$.

To compare the two star-products we have to do some calculation in the standard inhomogeneous complex co-ordinates $v^i := z^i/z^0, 1 \leq i \leq n$ on $\mathbb{C}\mathrm{P}^n$ First we see that the bidifferential operator $\tilde{N}$ can be written as $((u^1, \ldots, u^n)$ being another set of inhomogeneous co-ordinates on $\mathbb{C}\mathrm{P}^n$):

$$\tilde{N} = \left(1 + \sum_{j=1}^n v^j \bar{u}^j\right) \sum_{k,l=1}^n \left(v^k \bar{u}^l + \delta^{kl}\right) \frac{\partial}{\partial v^k} \frac{\partial}{\partial \bar{u}^l}.$$

For two locally defined real-analytic complex-valued functions ϕ, ψ on $\mathbb{C}\mathrm{P}^n$ we use their local holomorphic and antiholomorphic continuation to obtain: $(m \circ \tilde{N}(\phi,\psi))(v,\bar{v}) = m \circ \Delta_{u\bar{u}}\phi(u,\bar{v})\psi(v,\bar{u})$. For these kind of functions we can inductively prove the following local expression for the bidifferential operators $\tilde{M}_r$:

$$\tilde{M}_r(\phi,\psi)(v,\bar{v}) = m\circ\prod_{k=0}^{r-1}(\Delta_{u\bar{u}}+k(k-n))\phi(u,\bar{v})\psi(v,\bar{u}) =: m\circ p_r(\Delta)_{u\bar{u}}\phi(u,\bar{v})\psi(v,\bar{u})$$

Analogously we can conclude that the operators $\tilde{K}_r$ (24) can again be thought of as polynomials in the Laplacian which we denote by $\tilde{k}_r(\Delta)$. We thus have to prove the recursion formula [17, Prop. 1.i] for these polynomials:

$$(r+1)\tilde{k}_{r+1}(\Delta) - \left[\Delta\tilde{k}_r(\Delta) - \sum_{s=0}^{r-1}\frac{r!(r+3+s)}{(s+2)!(r-1-s)!}\tilde{k}_{r-s}(\Delta)\right] \stackrel{!}{=} 0$$

To prove this recursion formula we first write $\tilde{k}_r(\Delta) = \sum_{t=1}^{r}\frac{1}{t!}p_t(\Delta)A^{(t)}_{r-t}$ (see eqn. (15)) and get on the left hand side a linear combination of the $p_t(\Delta)$. Noting that the terms of $p_1(\Delta)$ and $p_{r+1}(\Delta)$ automatically cancel we obtain for the coefficient of the $p_s(\Delta)$ for $r \geq 2$ and $s = 2,\ldots,r$ the following expression:

$$A^{(s)}_{r+1-s}+\frac{1}{r+1}\sum_{t=s}^{r}\left[\binom{r+1}{t-1}+\binom{r}{t-1}\right]A^{(s)}_{t-s}-\frac{s}{r+1}\left(A^{(s-1)}_{r+1-s}-(s-1)A^{(s)}_{r-s}\right). \tag{25}$$

The polynomials $p_t(\Delta)$ are all linear independent therefore each of the coefficients in (25) has to vanish. In order to prove that the expression (25) vanishes for $r \geq 2$ and $s = 2,\ldots,r$ we first note that in the case $s = r$ it can be checked directly. Using the identity $A^{(s)}_{t+1-s} = A^{(s-1)}_{t+1-s} - sA^{(s)}_{t-s}$ for $s \geq 1$, $t \geq s$ we can prove that (25) vanishes by induction on r.

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